Electricity and Magnetism (WPPH 1600.2016-2017.2)

Exam 6, 21/06 2017, 9:00-12:00

Exam drafted by (name first examiner) Maxim S. Pchenitchnikov
Exam reviewed by (name second examiner) Steven Hoekstra
Grading scheme: Final Grade = 1 + 9x(Points scored/Max points)

5 questions; **with answers**

Name: ________________________________ Student number: ____________

This is an open question exam.
Write your name and student number on each sheet.
Clearly write your answers in the designated areas.
Write down your arguments and the intermediate steps in your calculations.
Use additional pages if the designated area is too short but **clearly indicate it.**
Use of a (graphical) calculator is allowed.
You may make use of the formula sheet (provided separately).
The same notation is used as in the book and lectures, i.e. a bold-face **A** is a vector, T is a scalar etc.
When you are asked to find a vector, do not forget about its direction!

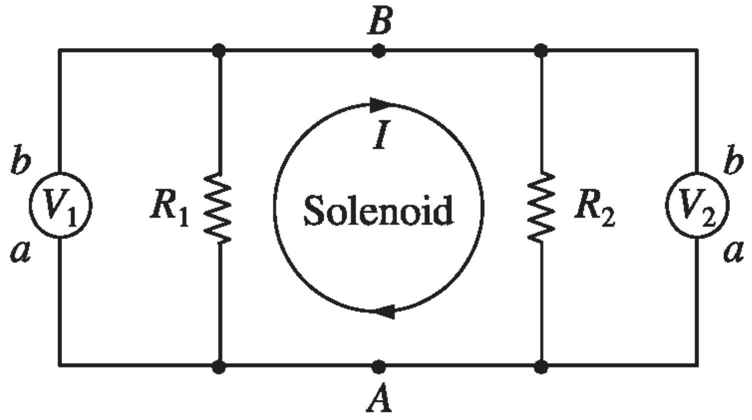
For administrative purposes only; do not fill!

The number of points at each (sub)question:

Question	**Max points**	**Points scored for sub-questions**					**Points scored**	**Test on**
		A	**B**	**C**	**D**	**E**		
1	10					--		emf
2	15					--		radiation
3	15							EM waves
4	15							Relativity
5	20							Potentials
Total	75							

Final grade: _________

Question 1 (10 points)



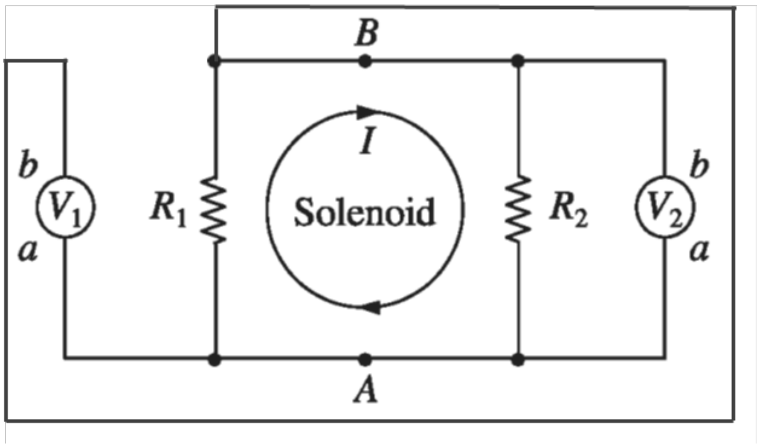
The current in a long solenoid is increasing linearly with time, so the flux is proportional to time t: $\Phi = \alpha t$. Two voltmeters are connected to diametrically opposite points *(A* and *B),* together with resistors (R_1 and R_2), as shown in the figure above. Assume that these are *ideal* voltmeters that draw negligible current (they have huge internal resistance), the magnetic field is restricted to the area of the solenoid, and the resistance of the connecting wires is zero.

A. In the diagram, show the direction of the induced current (if any), and explain why (2 points)

B. Calculate the value of the current *I* (if any) through R_1 and R_2 (2 points)

C. What is the reading on each voltmeter? (4 points)

D. Now the first voltmeter is rewired as shown in the figure below. How do their readings change? (2 points)



Answer to Question 1 **(Problem 7.53, Griffiths; considered at tutorials)**

A. By Lenz's law, the current flows counterclockwise (2 points)

B. $\varepsilon = -\frac{d\Phi}{dt} = -\alpha;$ (1 point)

$I = \frac{\alpha}{R_1+R_2}$ (1 point)

C. $V_1 = IR_1 = \frac{\alpha R_1}{R_1+R_2}$ (V_b is the higher potential) (2 points)

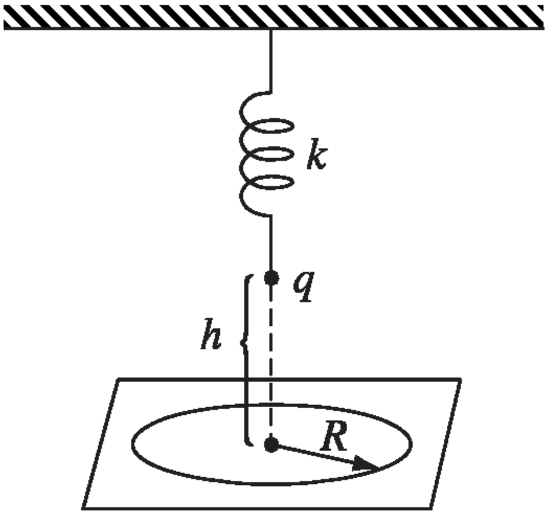
(-1 point if the sign is wrong)

$V_2 = -IR_2 = \frac{-\alpha R_2}{R_1+R_2}$ (V_b is lower) (2 points)

(-1 point if the sign is wrong)

D. $V_1 = V_2$ (2 points)

Question 2 (15 points)



A particle of mass *m* and charge *q* is attached to a spring with force constant *k,* hanging from the ceiling (see Figure). Its equilibrium position is a distance *h* above the floor. It is pulled down a distance *d* below equilibrium and released, at time *t* = 0. Consider the *z*-axis in the vertical direction and neglect all mechanical frictions involved.

A. Write down the oscillating (i.e. as a function of time *t*) electric dipole moment **p** and its frequency. (3 points)

B. Write down the averaged Poynting vector of such a dipole. (2 points)

C. Under the usual assumptions $d \ll \lambda \ll h,$ calculate the intensity I_f of the radiation hitting the floor, as a function of the distance *R* from the point directly below *q. (Note:* The intensity here is the average power per unit area of *floor).* Neglect the radiative damping of the oscillator. (8 points)

D. Why will the amplitude of oscillations decrease in time? (2 points)

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<u>Answer to Question 2 (Problem 11.22 Griffiths; considered at tutorials)</u>

A.

$\mathbf{p} = qz(t)\,\hat{\mathbf{z}}$ (1 point; if no $\hat{\mathbf{z}}$, 0.5 points)

$\mathbf{p} = qd\cos(\omega t)\,\hat{\mathbf{z}} = p_0\cos(\omega t)\,\hat{\mathbf{z}}$ (1 point, or 2 points if written right away; if no $\hat{\mathbf{z}}$, 0.5 points)

$\omega = \sqrt{k/m}$ (1 point)

B.

$\langle\mathbf{S}\rangle = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\hat{\mathbf{r}}$ (from the formula sheet; 1 point)

$= \frac{\mu_0 q^2 d^2 \omega^4}{32\pi^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\hat{\mathbf{r}}$ (1 point)

C. Power per unit area of floor

$I_f = \langle\mathbf{S}\rangle\cdot\hat{\mathbf{z}} = \frac{\mu_0 q^2 d^2 \omega^4}{32\pi^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\cos\theta$ (3 points)

$\sin\theta = \frac{R}{r};\ \cos\theta = \frac{h}{r};\ r^2 = R^2 + h^2$ (3 point)

$I_f = \frac{\mu_0 q^2 d^2 \omega^4}{32\pi^2 c}\frac{R^2}{R^2+h^2}\left(\frac{1}{R^2+h^2}\right)\frac{h}{\sqrt{R^2+h^2}}$ (2 points)

$= \frac{\mu_0 q^2 d^2 \omega^4}{32\pi^2 c}\frac{R^2 h}{(R^2+h^2)^{\frac{5}{2}}}$ (the same 2 points)

$(var) = \frac{\mu_0 q^2 d^2 k^2}{32\pi^2 c m^2}\frac{R^2 h}{(R^2+h^2)^{5/2}}$ (the same 2 points)

D. The amplitude of the oscillation will gradually decrease because the dipole is losing energy in the form of radiation. (2 points)

Question 3 (15 points)

A plane electromagnetic wave $E_0 cos(kz - \omega t)$ travelling through vacuum in the positive z direction and polarized into the x direction, encounters a perfect conductor, occupying the region $z \geq 0$, and reflects back. The electric field inside a perfect conductor is zero.

A. Find the complete electric field of the plane electromagnetic wave in the $z < 0$ region, by invoking the proper boundary condition (see the formula sheet). (5 points)

You might find useful the following relation: $cos\alpha - cos\beta = -2sin\frac{\alpha+\beta}{2}sin\frac{\alpha-\beta}{2}$

B. Draw schematically the electric field as a function of z for a few periods for $\omega t = -\pi/2, 0, +\pi/2$. (1.5 points)

C. Find the accompanying magnetic field in the $z < 0$ region. (5 points)

You might find useful the following relation: $cos\alpha + cos\beta = 2cos\frac{\alpha+\beta}{2}cos\frac{\alpha-\beta}{2}$

D. Draw schematically the magnetic field as a function of z for a few periods for $\omega t = 0, \pi/2, \pi$. (1.5 points)

E. What is the phase shift between the electric and magnetic fields? (2 points)

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Answer to Question 3 (Griffiths, Problem 9.34 modified)

A. Because the EM wave orthogonal to the interface, the boundary condition

$\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0$

$\mathbf{E}_2^{\parallel} = 0$ because the conductor is perfect (1 point)

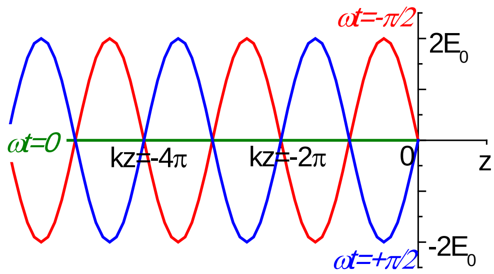
$\mathrm{E_I} + \mathrm{E}_R = 0$; $\mathrm{E_R} = -\mathrm{E}_I$ - the reflected wave has a π phase shift (1 point)

$\mathbf{E} = E_0[cos(kz-\omega t) - cos(kz+\omega t)]\hat{\mathbf{x}}$ (2 points)

(-1 point if no $\hat{\mathbf{x}}$)

$\mathbf{E} = 2E_0 sin(kz)sin(\omega t)\ \hat{\mathbf{x}}$ (it's a standing wave) (1 point)

B. (1.5 points; 0.5 per curve)



NB: No penalty if the signs are flipped or no tick labels. Important to draw *0* at *z=0*.

C.

$\mathbf{B} = \frac{E_0}{c}[cos(kz-\omega t) + cos(kz+\omega t)]\hat{\mathbf{y}}$ (4 points)

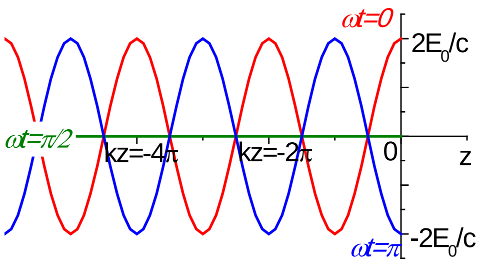
$B_0 = \frac{E_0}{c}$ because of scaling of the magnetic field (-1 point if incorrect)

$\hat{\mathbf{y}}$ because of polarization along $\hat{\mathbf{x}}$ and propagation along $\hat{\mathbf{z}}$ (-1 point if incorrect)

The "+" sign because $\mathbf{E}$ changes the sign upon reflection so $\mathbf{B}$ does not, and $\mathbf{E} \times \mathbf{B}$ is directed to the propagation direction (-1 point if incorrect)

$\mathbf{B} = 2\frac{E_0}{c} cos(kz)cos(\omega t)\ \hat{\mathbf{y}}$ (it's a standing wave) (1 point)

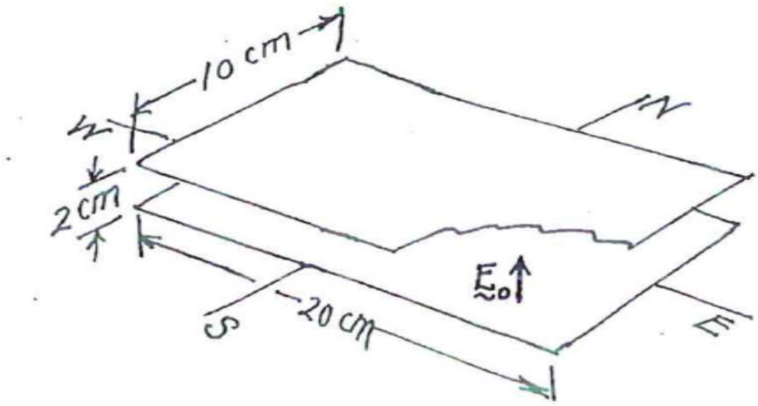
D. (0.5 points; 0.5 per curve)



NB: No penalty if the signs are flipped or no tick labels. Important to draw *non-zero* at *z=0*.

E. The phase shift is $\pi/2$ (or $-\pi/2$) (2 points)

Question 4 (15 points)



A capacitor consists of two parallel rectangular plates with a vertical separation of $d = 2\ cm$ (see the figure above). The East-West (EW) dimension of the plates is $a = 20\ cm$, the North-South (NS) dimension is $b = 10\ cm$. The capacitor has been charged by connecting it temporarily to a battery of $V = 300\ V$. The direction of the electric field vector is pointing upwards.

The capacitor is at rest in the laboratory system of coordinates. Consider an observer in a frame that is moving Eastward with respect to the laboratory frame, with speed $0.6c$. Provide the following quantities as they would be measured by the moving observer:

A. The three dimensions of the capacitor (3 points)

B. Change in the number of excess electrons on the negative plate (2 points)

C. The electric field strength between the plates (4 points)

D. The magnetic field between the plates (4 points)

E. What would be the electric field strength between the plates for a frame of a reference that is moving up with speed $0.6c$? (2 points)

Answers to Question 4

A. In the EW direction, the capacitor will experience Lorentz contraction with

$\beta = 0.6$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}} = 1.25$ (1 point)

The EW size becomes $\bar{a} = \frac{a}{\gamma} = \frac{20}{1.25} = 16\ cm$ (1 point)

NS and spacing remain the same. (1 point)

B. None – charge is invariant. (2 point)

C. $\bar{E} = E_0\gamma$ (1 point)

$E_0 = \frac{V}{d}$ (1 point)

$= \frac{300}{0.02} = 15000\ {}^{V}/_{m} = 15\ {}^{KV}/_{m}$ (1 point)

$\bar{E} = E_0\gamma = 18.75\ {}^{KV}/_{m}$ (1 point)

D. $\bar{\mathbf{B}} = -\frac{1}{c^2}(\mathbf{v} \times \bar{\mathbf{E}})$ (2 points)

so the magnetic field is N-S directed (1 points)

NB: if no direction, 3 points above are not awarded

$\bar{B} = \frac{v}{c^2}\bar{E} = \frac{0.6\ 18.75}{3\ 10^8} = 37.5 \cdot 10^{-8}\ T$ (1 point)

E. $\bar{E} = E_0$ (2 points)

Question 5. (20 points)

The scalar and vector potentials are given as follows:

$V = 0$

$$\mathbf{A} = \begin{cases} \frac{\mu_0 k}{4c}(ct - |x|)^2 \hat{\mathbf{z}}, & for\ |x| < ct \\ \mathbf{0}, & for\ |x| > ct \end{cases}$$

where k is a constant, $c = 1/\sqrt{\mu_0 \epsilon_0}$ (as usual), and $\mu = \mu_0, \epsilon = \epsilon_0$ everywhere.

A. Find and depict schematically (as a function of the respective coordinate) the electric field. (2 points)

B. Find and depict schematically (as a function of the respective coordinate) the magnetic field. (5 points)

C. Find the volume charge and current distributions that would give rise to the potentials (8 points)

D. Find the surface current density $\mathbf{K}$ (tip: use the formula sheet for the respected boundary condition) (4 points)

E. Explain relationship between $\mathbf{K}$ and the fields from the viewpoint of relativity theory (1 point)

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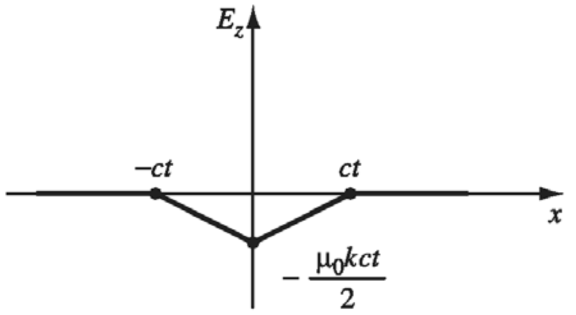
Answers to Question 5 (Griffiths, Example 10.1)

A. $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\frac{\partial \mathbf{A}}{\partial t}$ (1 point)

$= -\frac{\mu_0 k}{2}(ct - |x|)\ \hat{\mathbf{z}}, for\ |x| < ct$ (1 point)

$\mathbf{E} = 0, for\ |x| > ct$



(2 points)

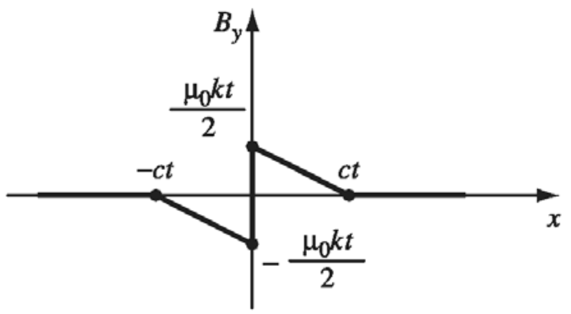
B. $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ (1 point)

$$= \frac{\mu_0 k}{4c}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & \frac{\partial}{\partial z} \\ 0 & 0 & (ct-|x|)^2 \end{vmatrix} = -\frac{\mu_0 k}{4c}\frac{\partial}{\partial x}(ct - |x|)^2\ \hat{\mathbf{y}}$$ (2 points)

$$= \begin{cases} +\frac{\mu_0 k}{2c}(ct - |x|)\ \ \hat{\mathbf{y}}, for\ x > 0 \\ -\frac{\mu_0 k}{2c}(ct - |x|)\ \ \hat{\mathbf{y}}, for\ x < 0 \end{cases}, for\ |x| < ct$$ (2 points)

$\mathbf{B} = 0, for\ |x| > ct$

No points are deducted if only the x>0 or x<0 solution is given



(2 points)

C. $\boldsymbol{\nabla} \cdot \mathbf{E} = {}^{\rho}/_{\epsilon}$ (1 point)

$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{dE_z}{dz} = 0, i.e.\ \rho = 0$ (2 points)

$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t};\ \mu_0 \mathbf{J} = \boldsymbol{\nabla} \times \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ (1 point)

$$\boldsymbol{\nabla} \times \mathbf{B} = \pm\frac{\mu_0 k}{2c}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & \frac{\partial}{\partial z} \\ 0 & ct-|x| & 0 \end{vmatrix} = -\frac{\mu_0 k}{2c}\hat{\mathbf{z}}$$ (1 point)

$\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} = -\,\mu_0\epsilon_0\frac{\mu_0 k}{2}\frac{\partial}{\partial t}(ct - |x|)\ \ \hat{\mathbf{z}} = -\,\mu_0\epsilon_0\frac{\mu_0 kc}{2}\hat{\mathbf{z}} = -\frac{\mu_0 k}{2c}\hat{\mathbf{z}}$ (1 point)

$\mu_0\mathbf{J} = \mathbf{0}$; $\mathbf{J} = \mathbf{0}$ (2 points)

So that both charge and current densities are zero!

If the sign was mixed up, i.e. $J = -\frac{k}{c}$ or similar, only 1 point is deducted

Another way is through the inhomogeneous wave equation for the potentials

$\square^2 V = -\frac{1}{\epsilon_0}\rho$; $\rho = 0$ (3 points)

$\square^2\mathbf{A} = -\mu_0\mathbf{J}$; $\frac{\partial^2}{\partial t^2}A_z - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}A_z = \frac{\mu_0 k}{2c} - \mu_0\epsilon_0\frac{\mu_0 k}{2c}c^2 = 0$; $\mathbf{J} = \mathbf{0}$ (5 points)

If the sign was mixed up, i.e. $J = -\frac{k}{c}$ or similar, only 1 point is deducted

Finally, some used the 4-vector potential $\square^2 A^\mu = -\mu_0 J^\mu$ to show that $J^\mu = 0$ (8 + 2 bonus points)

D. B has a discontinuity at $x = 0$ so that we should look at the respective boundary condition.

$\frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}$ (1 point)

$\frac{1}{\mu_0}\mathbf{B}_{x=+0} - \frac{1}{\mu_0}\mathbf{B}_{x=-0} = \mathbf{K}\times\hat{\mathbf{x}}$ (1 point)

$kt\ \hat{\mathbf{y}} = \mathbf{K}\times\hat{\mathbf{x}}$; $\mathbf{K} = kt\ \hat{\mathbf{z}}$ (2 points)

(-1 point for the none-$\hat{\mathbf{z}}$ direction)

(no penalty for the wrong sign)

-- a uniform surface current flowing in the *z* direction over the plane x = 0, which starts up at *t* = 0, and increases in proportion to *t.*

E. K starts up at *t* = 0. The news travels out (in both directions) at the speed of light: for points $|x| > ct$ the message ("current is now flowing") has not yet arrived, so the fields are zero. (2 point)

Maxim S. Pchenitchnikov **Steven Hoekstra**